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The Cantor Construction

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A theorem of Cantor states that no transformation maps a set onto the class of all its subsets. But it is possible for a transformation to map a set onto the class of its subsets of smaller cardinality. The aim of the paper is to call attention to the interesting properties of such sets and transformations.

The main result is a generalization of the Cantor construction of a set which is not in the range of a transformation which maps a set into the class of its subsets.

THEOREM 1. *Assume that a transformation T maps a set S into the class of its subsets. For each positive integer r, let S_r be the set of elements c of S for which no elements $c = c_0, c_1, \ldots, c_r = c$ of S can be chosen so that c_n belongs to Tc_{n-1} for $n = 1, \ldots, r$. Let S_∞ be the set of elements c of S for which no elements c_n of S can be chosen for every integer n so that $c_0 = c$ and c_n always belongs to Tc_{n-1}. Then none of the sets S_r, r finite or infinite, is in the range of T.*

The theorem has an interesting consequence in the case that r is infinite.

THEOREM 2. *If a set S has the same cardinality as the class of its subsets of smaller cardinality, then a one-to-one transformation T exists of S onto the class of its subsets of smaller cardinality such that no elements c_n of S can be chosen for every integer n so that c_n always belongs to Tc_{n-1}. If T_+ and T_- are two such transformations, a unique one-to-one transformation W exists of S onto itself such that, for every element a of S, $T_+ Wa$ is the set of elements of the form Wb with b in $T_- a$.*

The theorem allows a partial ordering to be defined on the set.

THEOREM 3. *Assume that a one-to-one transformation T maps a set S onto the class of its subsets of smaller cardinality in such a way that no elements c_n of S can be chosen for every integer n so that c_n always belongs to Tc_{n-1}. Then a partial ordering of S exists such that $a < b$ if, and only if,*

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elements $b = c_0, c_1, \ldots, c_r = a$ of S exist for some positive integer r so that c_n belongs to Tc_{n-1} for $n = 1, \ldots, r$. Every totally ordered subset of S is well-ordered. A subset of S is of smaller cardinality if, and only if, it has an upper bound in S. If a set S_- is of smaller cardinality than S, then the cardinality of the class of all subsets of S_- is less than or equal to the cardinality of S.

A stronger conclusion is obtained from a stronger hypothesis.

THEOREM 4. *Assume that a one-to-one transformation T maps a set S onto the class of its subsets of smaller cardinality in such a way that no elements c_n of S be chosen for every integer n so that c_n always belongs to Tc_{n-1}. If S does not have the same cardinality as the class of all subsets of a set, then a well-ordering of S exists such that $a < b$ whenever a belongs to Tb and such that every bounded subset of S is of smaller cardinality.*

A construction of such sets is easily made.

THEOREM 5. *An infinite set S has the same cardinality as the class of its subsets of smaller cardinality if it has the same cardinality as the class of all subsets of a set and if, for each set S_- of smaller cardinality than S, the cardinality of the class of all subsets of S_- is less than or equal to the cardinality of S.*

THEOREM 6. *A set has the same cardinality as the class of its subsets of smaller cardinality if it is not a union of smaller cardinality than S of sets of smaller cardinality than S and if, for each set S_- of smaller cardinality than S, the cardinality of the class of all subsets of S_- is less than the cardinality of S.*

An Ulam measure for a set S is a function σ with values zero and one, defined on all subsets of S, such that the value of σ on any countable union of sets is the maximum of its values on these sets and the value of σ on any countable intersection of sets is the minimum of its values on these sets. An Ulam measure σ is said to be trivial if an element s of S exists such that σ is one on sets which contain s and is zero otherwise. A theorem of Ulam [1] reduces the determination of Ulam measures to a situation in which the Cantor construction is applicable.

THEOREM 7. *If S is a set of least cardinality which admits a nontrivial Ulam measure σ, then S is an uncountable set which has the same cardinality as the class of its subsets of smaller cardinality and which does not have the same cardinality as the class of all subsets of a set. The value of*

σ on any union of smaller cardinality than S of subsets of S is the maximum of its values on these sets and the value of σ on any intersection of smaller cardinality than S of subsets of S is the minimum of its values on these sets.

Note that it is consistent with the axioms of set theory to assume than an uncountable set which has the same cardinality as the class of its subsets of smaller cardinality has the same cardinality as the class of all subsets of a set. For if any such set exists, the set can be chosen so that the transformation T of Theorem 2 is the identity transformation. Then the class of subsets of smaller cardinality is closed under all the usual axioms of set theory. There is no way of having the existence of larger cardinalities without introducing a new axiom in set theory saying that it does. It is always consistent with the axioms of set theory to assume that no nontrivial Ulam measure exists.

Other axioms in set theory are suggested by the Cantor construction. A natural generalization of the general continuum hypothesis is to assume that a set S which has the same cardinality as the class of all subsets of a set has the same cardinality as the class of its subsets of smaller cardinality. A stronger condition is to assume the existence of a set S_- of greatest cardinality such that S has the same cardinality as the class of all subsets of S_-. These conditions are of particular interest when S is the class of all subsets of a countably infinite set.

Proof of Theorem 1. Consider the set S_r for a positive integer r. If c is an element of S which does not belong to S_r, then elements $c = c_0, c_1, \ldots, c_r = c$ of S exist such that c_n belongs to Tc_{n-1} for $n = 1, \ldots, r$. Since none of the elements c_n belong to S_r, Tc_n is never equal to S_r. In particular, Tc is not equal to S_r. But if c is an element of S which does belong to S_r, then c does not belong to Tc because $c_n = c$ for $n = 0, \ldots, r$ is then a choice of elements of S such that c_n belongs to Tc_{n-1} for $n = 1, \ldots, r$. So Tc is again not equal to S_r. A similar argument shows that S_∞ is not in the range of T.

Proof of Theorem 2. Consider any one-to-one transformation T of S onto the class of its subsets of smaller cardinality. Define S_∞ as in Theorem 1. The conclusion of the theorem is obvious in the case that S contains less than two elements. If it contains more than one element, then every one-element set is in the range of T. Since S_∞ is not in the range of T, it has the same cardinality as S. If c is an element of S which does not belong to S_∞, then Tc is not a subset of S_∞ because it contains an element which does not belong to S_∞. Since every subset of S_∞ of smaller cardinality is in the range of T, it is of the form Tc for an element c of S_∞. The choice of T can clearly be made so that $S_\infty = S$. Assume that T is so chosen.

If T_+ and T_- are two such transformations, let $\mathscr{C}$ be the class of subsets

C of S, such that Tc is contained in C whenever c belongs to C, with this property: A unique one-to-one transformation W of C into S exists such that, whenever a is an element of C such that $T_- a$ is contained in C, then $T_+ Wa$ is the set of elements of the form Wb with b in $T_- a$. For example, the empty set is of class $\mathscr{C}$. The union of every nonempty well-ordered subclass of $\mathscr{C}$ is clearly a set of class $\mathscr{C}$. By Zorn's lemma, the class $\mathscr{C}$ contains a maximal set C. If c is any element of S such that $T_- c$ is contained in C, then c belongs to C.

Argue by contradiction, assuming that some element c of S does not belong to C. Then a sequence of elements c_n of S can be defined inductively for nonnegative integers n so that $c_0 = c$ and c_n is an element of Tc_{n-1} which does not belong to C, for every positive integer n. Since this contradicts the construction of T_-, C contains every element of S. A similar argument shows that W maps S onto itself.

Proof of Theorem 3. The properties of a partial ordering are easily verified since no elements $c_0, c_1, \ldots, c_r$ of S exist for a positive integer r so that $c_0 = c_r$ and c_n belongs to Tc_{n-1} for $n = 1, \ldots, r$. To show that every totally ordered subset C of S is well-ordered, it is sufficient to show that no elements c_n of C can be chosen for every nonnegative integer so that $c_n < c_{n-1}$ when n is positive. This is true by the definition of inequality and the hypotheses on the transformation T.

It will be shown that no subset of S of equal cardinality is a union of smaller cardinality of sets of smaller cardinality. It is clearly sufficient to show that S is not a union of smaller cardinality of sets of smaller cardinality. Argue by contradiction, assuming that it is. Then a subset C of S of smaller cardinality exists such that the given sets are of the form Tc with c in C. An element a of S exists such that $Ta = C$. Since the union of the sets Tc with c in C is all of S, a belongs to Tb for an element b of C. Define $c_n = a$ for even integers n and $c_n = b$ for odd integers n. A contradiction is obtained because c_n belongs to Tc_{n-1} for every integer n.

Every subset C of S of smaller cardinality has as an upper bound the unique element a of S such that $Ta = C$. Argue by contradiction, assuming that some subset of S of equal cardinality has an upper bound c in S. An element c_n of S can be chosen inductively for every nonnegative integer n so that $c_0 = c$, so that c_n belongs to Tc_{n-1} for every positive integer n, and so that c_n is an upper bound of a subset of S of equal cardinality. The choices are possible because Tc_n is always a subset of S of smaller cardinality and because the set of elements of S which are less than c_n is the union of the sets of elements of S which are less than or equal to b for elements b of Tc_n. This gives the desired contradiction.

If S_- is a subset of S of smaller cardinality, let S_+ be the set of elements s of S such that Ts is a subset of S_-. Since T acts as a one-to-one transformation of S_+ onto the class of all subsets of S_-, the cardinality of the class of all subsets of S_- is less than or equal to the cardinality of S.

Proof of Theorem 4. Let $\mathscr{C}$ be the class of well-orderings of subsets C of S with these properties: When c belongs to C, Tc is contained in the set of elements of C which are less than c. When a is an element of C and b is either an element of C such that $a < b$ or an element of S not in C such that Tb is contained in C, then the least upper bound of Ta is less than or equal to the least upper bound of Tb and, if these least upper bounds are equal, the ordinal number of Ta is less than or equal to the ordinal number of Tb. (The least upper bounds are taken in a larger well-ordered set if they do not exist in C.)

A well-ordering of a set A is considered less than or equal to a well-ordering of a set B if A is contained in B, if the ordering of A is the restriction of the ordering of B, and if every element of B which is less than an element of A is an element of A.

The well-ordering of the empty set is of class $\mathscr{C}$. The least upper bound of any nonempty well-ordered class of well-orderings of class $\mathscr{C}$ is a well-ordering of class $\mathscr{C}$. By Zorn's lemma, a maximal well-ordering of class $\mathscr{C}$ exists. It will be shown that the set C on which a maximal well-ordering of class $\mathscr{C}$ is defined contains every element of S.

Argue by contradiction, assuming that some element of S does not belong to C. Then by the proof of Theorem 2, an element c of S exists, which does not belong to C, such that Tc is a subset of C. Choose such an element c so as to minimize the least upper bound of Tc and the ordinal number of Tc among sets of smallest least upper bound. Then the set C' obtained by adjoining c to C admits a well-ordering of class $\mathscr{C}$ in which c is an upper bound of C.

It remains to show that a well-ordering of S which is of class $\mathscr{C}$ has the desired properties. Argue by contradiction, assuming that some bounded subset of S is of equal cardinality. Then a least element c of S exists such that the set of elements which are less than c has the same cardinality as S. Since a set of the same cardinality as S is not a union of smaller cardinality of sets of smaller cardinality by (the proof of) Theorem 3, every subset of S of smaller cardinality which has c as a bound has a bound less than c. So the least upper bound a of Tc is less than c. By construction, the set S_- of elements of S which are less than or equal to a is of smaller cardinality than S. By Theorem 3, the class of all subsets of S_- is of smaller cardinality than S. A contradiction is now obtained since T acts as a one-to-one transformation of the set of elements of S which are less than c into the class of subsets of S_-. The theorem follows.

Proof of Theorem 5. By hypothesis, S has the same cardinality as the class of all subsets of some set C of cardinality α. The cardinality of S is equal to the cardinality of the set F of all functions defined on C with values zero and one. The cardinality of the class of all subsets of S of cardinality at most α is less than or equal to the cardinality of the set of all functions defined on C with values in F. But every function defined on C with values in F can be regarded as a function defined on the Cartesian product $C \times C$ with values in the set consisting of zero and one. Since C is an infinite set, $C \times C$ has the same cardinality as C. It follows that the cardinality of the class of all subsets of S of cardinality at most α is less than or equal to the cardinality of S. Since the reverse inequality clearly holds, the two cardinalities are equal.

Consider any one-to-one transformation T of S onto the class of its subsets of cardinality at most α. Since the set S_∞ defined by Theorem 1 does not belong to the range of T, it contains a subset S_- of cardinality α. Then the set S_+ of elements of S such that Ts is contained in S_- has the same cardinality as the class of all subsets of a set of cardinality α. Since S_+ is contained in S_∞ by the proof of Theorem 2, the cardinality of S is less than or equal to the cardinality of S_∞. Since the reverse inequality clearly holds, S_∞ has the same cardinality as S. As in the proof of Theorem 2, the transformation T can be chosen so that S_∞ contains every element of S.

The proof of the theorem is complicated by the possibility that many cardinal numbers α may exist such that S has the same cardinality as the class of all subsets of a set of cardinality α. For each such cardinal number α, a set S_α of the same cardinality as S exists and there exists a one-to-one transformation T_α of S onto the class of its subsets of cardinality at most α such that no elements c_n of S_α can be chosen for every integer n so that c_n always belongs to $T_\alpha c_{n-1}$. If α and β are any two such cardinal numbers, α less than β, then by the proof of Theorem 2 a unique one-to-one transformation W (which depends on α and β) exists which maps S_α into S_β in such a way that, for every element a of S_α, $T_\beta Wa$ is the set of elements of the form Wb with b in $T_\alpha a$. With no loss of generality, it can be assumed (by an inductive construction) that the sets and transformations are chosen so that W is always an inclusion of S_α in S_β.

The set of cardinal numbers α such that S has the same cardinality as the class of all subsets of a set of cardinality α is less than or equal to the cardinality of S. Since the cardinality of the Cartesian product $S \times S$ is equal to the cardinality of S, the cardinality of the union of the sets S_α is equal to the cardinality of S. The sets and transformations can therefore be chosen so that S is equal to the union of the sets S_α. Let T be the unique transformation of S into the class of its subsets of smaller cardinality which extends each of the transformations T_α. If a cardinal number γ is less than the cardinality of S, then the cardinality of the class of all subsets of a set of cardinality γ is less than or equal to the cardinality of S by hypothesis. It follows that γ is less than or equal to α for some cardinal number α such that

S has the same cardinality as the class of all subsets of a set of cardinality α. So every set whose cardinality is less than or equal to γ is in the range of T. By the arbitrariness of γ, T maps S onto the class of its subsets of smaller cardinality. Since T is one-to-one, S has the same cardinality as the class of its subsets of smaller cardinality.

Proof of Theorem 6. A well-ordered class $\mathscr{C}$ of subsets of S of smaller cardinality can be chosen so that every subset of S of smaller cardinality has the same cardinality as some set of class $\mathscr{C}$ and so that the union of the sets of class $\mathscr{C}$ is S. Since S is not a union of smaller cardinality of sets of smaller cardinality, every subset of S of smaller cardinality is contained in some set of class $\mathscr{C}$. By hypothesis the cardinality of the class of all subsets of S_- is less than the cardinality of S if S_- is a set of class $\mathscr{C}$. It follows that the cardinality of the class of all subsets of S of smaller cardinality is less than or equal to the cardinality of S. Since the reverse inequality clearly holds, the two cardinalities are equal.

Proof of Theorem 7. Since the set S admits a nontrivial Banach measure σ, it is uncountable. Since S is a set of least cardinality which admits a nontrivial Banach measure, every function which maps S into a set of smaller cardinality maps σ into a trivial Banach measure.

Consider any nonempty class $\mathscr{C}$ of subsets of S which is of smaller cardinality than S. It will be shown that the value of σ on the union of the sets of class $\mathscr{C}$ is equal to the maximum of its values on these sets. This is clearly the case if an equivalence relation exists on S such that each set of class $\mathscr{C}$ is a union of equivalence clases and such that the quotient set is of smaller cardinality. Such an equivalence relation always exists when the class $\mathscr{C}$ is well-ordered.

The desired conclusion is easily obtained if σ is one on some set of class $\mathscr{C}$. Otherwise it must be obtained when σ is zero on every set of the class. Let $\mathscr{C}'$ be the class of unions of sets of class $\mathscr{C}$ on which σ is zero. Since every well-ordered subclass of $\mathscr{C}'$ is of smaller cardinality than S, the union of the subclass is of class $\mathscr{C}'$. By Zorn's lemma,the class $\mathscr{C}'$ contains a maximal set. Since each set of class $\mathscr{C}$ is of class $\mathscr{C}'$, a maximal set of class $\mathscr{C}'$ contains every set of class $\mathscr{C}$.

A similar argument shows that the value of σ on the intersection of the sets of class $\mathscr{C}$ is the minimum of its values on these sets. Since σ is zero on finite sets, it is zero on any set of smaller cardinality than S. Since σ is one on S, S is not a union of smaller cardinality than S of sets of smaller cardinality than S.

It will be shown that no set S_- of smaller cardinality than S exists such that the cardinality of S is less than or equal to the cardinality of the class of all subsets of S_-. Argue by contradiction, assuming that such a set S_- exists. Then a function $\langle a, b\rangle$ of elements a of S and b of S_- exists, with values zero and one, such that any two elements a_1 and a_2 of S are equal whenever $\langle a_1, b\rangle$ and $a_1, b\rangle$ are equal for every element b of S_-.

Let C be the set of elements b of S_- such that σ has value one on the set of elements a of S such that $\langle a, b\rangle = 1$. Then for every element b of S_- which does not belong to C, σ has value one on the set elements a of S such that $\langle a, b\rangle = 0$. Since the cardinality of S_- is less than the cardinality of S, σ has value one on the set of elements a of S such that $\langle a, b\rangle = 1$ whenever b is in C and such that $\langle a, b\rangle = 0$ whenever b is not in C. Since σ has value zero on the empty set, such an element a exists, and it is unique by the properties of the pairing between S and S_-. This contradicts the hypothesis that σ is nontrivial since it has value one on a one-element set.

It has been shown that for every set S_- of smaller cardinality than S, the cardinality of the class of all subsets of S_- is less than the cardinality of S. The theorem follows from Theorem 6.

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